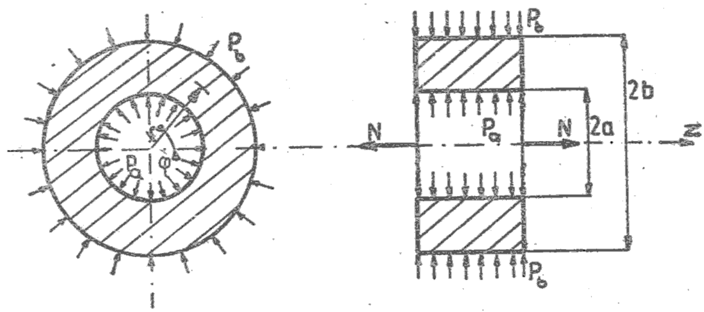
INSTYTUT MECHANIKI
I PODSTAW KONSTRUKCJI MASZYN
WYDZIAŁU MECHANICZNEGO

Kołowe cylindry grubościenne w stanie sprężystym

1. Zagadnienie Lame



Rys. 1.

Dane :	a = const	Znaleźć :
	b = const	rozkład przemieszczeń
	p_a = const	" odkształceń
	p_b = const	" naprężeń

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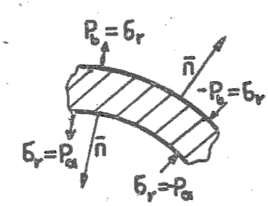
r.r.w. (związek statyczny)

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} + R = 0 \quad \left| \begin{array}{l} \sigma_r = ? \\ \sigma_\varphi = ? \end{array} \right.$$

w.b.

$$\sigma_r(a) = \pm p_a$$

$$\sigma_r(b) = \pm p_b$$



rys. 2.

$R = \frac{\gamma}{g}\omega^2 r$

Z : 1°) $R = 0$ (cylinder nieruchomy)

2°) $\varepsilon_z = const$ (rura dostatecznie długa, efektów brzegowych nie rozpatruje się)

Metoda - sprowadzenie r.r.w. do niewiadomych przemieszczeń.

Prawo Hooke'a (związki fizyczne)

(2)
$$\begin{cases} \sigma_r = \dfrac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_r + \nu(\varepsilon_\varphi + \varepsilon_z)\right] \\ \sigma_\varphi = \dfrac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_\varphi + \nu(\varepsilon_r + \varepsilon_z)\right] \\ \sigma_z = \dfrac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_r + \varepsilon_\varphi)\right] \end{cases}$$

(związki geometryczne)

(3)
$$\varepsilon_r = \frac{du}{dr} = u'$$
$$\varepsilon_\varphi = \frac{u}{r}$$

$u = u(r)$
$v = 0$
$w = w(z)$

Podstawiamy do (2)

(4)
$$\begin{cases} \sigma_r = \dfrac{2G}{1-2\nu}\left[(1-\nu)u' + \nu\left(\dfrac{u}{r} + \varepsilon_z\right)\right] \\ \sigma_\varphi = \dfrac{2G}{1-2\nu}\left[(1-\nu)\dfrac{u}{r} + \nu(u' + \varepsilon_z)\right] \\ \sigma_z = \dfrac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_z + \nu\left(u' + \dfrac{u}{r}\right)\right] \end{cases}$$

$$(5)\begin{cases} \dfrac{d\sigma_r}{dr} = \dfrac{2G}{1-2\nu}\left[(1-\nu)u'' + \nu\left(\dfrac{u'r-u}{r^2} + 0\right)\right] \\ \sigma_r - \sigma_\varphi = \dfrac{2G}{1-2\nu}\left[(1-\nu)\left(u' - \dfrac{u}{r}\right) + \nu\left(\dfrac{u}{r} - u' + \varepsilon_z - \varepsilon_z\right)\right] = \end{cases}$$

$$= \frac{2G}{1-2\nu}(1-2\nu)\left(u' - \frac{u}{r}\right)$$

po podstawieniu do (1) i uporządkowaniu :

$$(1-\nu)u'' + (1-\nu)\frac{u'}{r} - (1-\nu)\frac{u}{r^2} = 0$$

$$(6) \qquad u'' + \frac{u'}{r} - \frac{u}{r^2} = 0 \longrightarrow u(r) = ?$$

$$= -\frac{\sigma_{\omega}^2 (1+\nu)(1-2\nu)}{9E\,(1-\nu)}\, r$$

(równanie różniczkowe Eulera/

$u(r) =$

I sposób :

Z :

$\bar{u}(r) = r^\alpha$

$u' = \alpha r^{\alpha-1}$

$u'' = \alpha(\alpha-1)r^{\alpha-2}$

po podst. do (6)

$$\alpha(\alpha-1)r^{\alpha-2} + \frac{\alpha r^{\alpha-1}}{r} - \frac{r^\alpha}{r^2} = 0$$

$\alpha^2 = 1$

$\alpha_1 = 1$

$\alpha_2 = -1$

II sposób :

$$\frac{u'}{r} - \frac{u}{r^2} = \frac{u'r - u}{r^2} = \frac{d}{dr}\left(\frac{u}{r}\right)$$

$$u'' + \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{d}{dr}\left(u' + \frac{u}{r}\right)$$

$$u' + \frac{u}{r} = \frac{1}{r}(u'r + u) =$$

$$= \frac{1}{r}\frac{d}{dr}(u\,r)$$

$$u'' + \frac{u'}{r} - \frac{u}{r^2} = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(u\,r)\right] = 0$$

$$\frac{1}{r}\frac{d}{dr}(u\,r) = \bar{C}_1 \quad \Big/ \cdot r \int$$

$$u_1(r) = C_1 r^{\alpha_1} = C_1 r \qquad ur = \frac{\bar{C}_1}{2} r^2 + C_2$$

$$u_2(r) = C_2 r^{\alpha_2} = C_2 r^{-1} \qquad u = \frac{\bar{C}_1}{2} r + \frac{C_2}{r}$$

$$u(r) = u_1(r) + u_2(r) \qquad \frac{\bar{C}_1}{2} = C_1$$

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(7) $$u(r) = C_1 r + \frac{C_2}{r}$$

po podstawieniu (7) do (3)

(8) $$\begin{cases} \varepsilon_r = C_1 - \dfrac{C_2}{r^2} \\ \varepsilon_\varphi = C_1 + \dfrac{C_2}{r^2} \end{cases} \rightarrow \varepsilon_\varphi + \varepsilon_r = 2C_1 = \text{const.}$$

naprężenia (2) po uwzględnieniu (8) są :

(9)

$$\sigma_r = \frac{2\,G}{1-2\nu}\left[C_1 - (1-2\nu)\frac{C_2}{r^2} + \nu\varepsilon_z\right] = A - \frac{B}{r^2}$$

$$\sigma_\varphi = \frac{2\,G}{1-2\nu}\left[C_1 + (1-2\nu)\frac{C_2}{r^2} + \nu\varepsilon_z\right] = A + \frac{B}{r^2}$$

gdzie : $A = \dfrac{2\,G}{1-2\nu}\left(C_1 + \nu\varepsilon_z\right)$

$B = 2G\,C_2$

oraz :

(10) $$\sigma_z = \frac{2\,G}{1-2\nu}\left[(1-\nu)\varepsilon_z + \nu 2C_1\right] = \text{const} \rightarrow \sigma_z = \frac{N}{F}$$

$A = ?$, $B = ?$

$C_1 = ?$, $C_2 = ?$

(wyznaczamy z warunków brzegowych) np.

$$\sigma_r(b) = -p_b$$

$$\sigma_r(a) = -p_a$$

(11) $\sigma_r(a) = -p_a = A - \frac{B}{a^2}$ i $\sigma_r(b) = -p_b = A - \frac{B}{b^2}$

stąd

(12)
$$\begin{cases} A = \dfrac{p_a a^2 - p_b b^2}{b^2 - a^2} & C_1 = \dfrac{1-2\nu}{2G} A - \nu\varepsilon_z \\ B = \dfrac{a^2 b^2 (p_a - p_b)}{b^2 - a^2} & C_2 = \dfrac{1}{2G} B \end{cases}$$

po podstawieniu za C_1, C_2 do (7)

ostatecznie otrzymamy :

przemieszczenie

(13) $$u(r) = \frac{1}{2G(b^2-a^2)}\left[(1-2\nu)(p_a a^2 - p_b b^2) r + a^2 b^2 (p_a - p_b)\frac{1}{r}\right] - \nu\varepsilon_z r$$

do (8)

odkształcenia

(14)
$$\varepsilon_r = \frac{1}{2G(b^2-a^2)}\left[(1-2\nu)(p_a a^2 - p_b b^2) - a^2 b^2 (p_a - p_b)\frac{1}{r^2}\right] - \nu\varepsilon_z$$

$$\varepsilon_\varphi = \frac{1}{2G(b^2-a^2)}\left[(1-2\nu)(p_a a^2 - p_b b^2) + a^2 b^2 (p_a - p_b)\frac{1}{r^2}\right] - \nu\varepsilon_z$$

do (10)

(15) $$\varepsilon_z = -\frac{2\nu}{E} A + \frac{1}{E}\sigma_z = -\frac{2\nu}{E}\,\frac{p_a a^2 - p_b b^2}{b^2 - a^2} + \frac{1}{E}\frac{N}{F} = \text{const}$$

$$u = \left[\frac{1-2\nu}{2G} A - \nu\varepsilon_z\right] r + \frac{B}{2G}\frac{1}{r}$$

$$u = C_1 r + \frac{C_2}{r}$$

$$C_1 = \frac{1-2\nu}{2G} A - \nu\varepsilon_z$$

$$C_2 = \frac{1}{2G} B$$

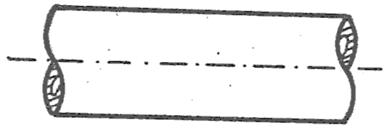
$$\varepsilon_z = -\frac{2\nu}{E} A + \frac{1}{E}\sigma_z$$

$$-\frac{(3+\nu)\gamma\omega^2 r^2}{8g}$$

$$-\frac{(1+3\nu)\gamma\omega^2 r^2}{8g}$$

$$\varepsilon_z = f(\sigma_z) = ?$$

a)

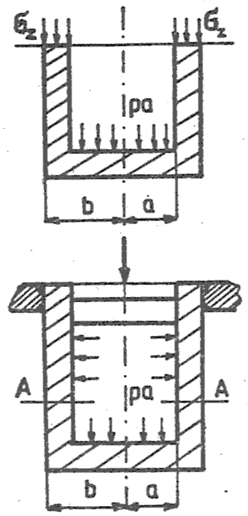


rys.3.

$$\varepsilon_z = 0$$

$$z(15) \rightarrow \quad \sigma_z = 2\nu \frac{a^2 p_a - b^2 p_b}{b^2 - a^2}$$

b)



rys. 4.

$$\sigma_z = \frac{N}{F}$$

$$N = p_a \pi a^2$$

$$F = \pi (b^2 - a^2)$$

$$\sigma_z = p_a \frac{a^2}{b^2 - a^2}$$

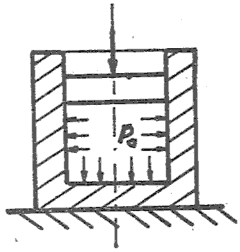
z(15) :

$$\varepsilon_z = -\frac{2\nu}{E} \frac{a^2 p_a - b^2 p_b}{b^2 - a^2} + \frac{1}{E} p_a \frac{a^2}{b^2 - a^2}$$

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)\gamma\omega^2 r^2}{8g}$$

$$\sigma_\varphi = A + \frac{B}{r^2} - \frac{(1+3\nu)\gamma\omega^2 r^2}{8g}$$

TARCZA WIRUJĄCA



$$\sigma_z = 0$$

$$\varepsilon_z = -\frac{2\nu}{E} \quad \frac{a^2 p_a - b^2 p_b}{b^2 - a^2}$$

rys.5.

naprężenia

(16)
$$\begin{cases} \sigma_r(r) = \dfrac{a^2 p_a - b^2 p_b}{b^2 - a^2} - \dfrac{a^2 b^2 (p_a - p_b)}{b^2 - a^2} \quad \dfrac{1}{r^2} \\[2ex] \sigma_\varphi(r) = \dfrac{a^2 p_a - b^2 p_b}{b^2 - a^2} + \dfrac{a^2 b^2 (p_a - p_b)}{b^2 - a^2} \quad \dfrac{1}{r^2} \\[2ex] \sigma_z \leftarrow z(15) \qquad \text{(zależne od przypadku)} \end{cases}$$

(17)
$$\begin{cases} \sigma_r(r) = A - \dfrac{B}{r^2} \\[2ex] \sigma_\varphi(r) = A + \dfrac{B}{r^2} \\[2ex] \sigma_z = \ldots\ldots\ldots\ldots \end{cases}$$

gdzie :

$$A = p_a \frac{n^2 - t}{1 - n^2} \qquad n^2 = \frac{a^2}{b^2} < 1$$

$$B = p_a a^2 \frac{1 - t}{1 - n^2} \qquad t = \frac{p_b}{p_a} \qquad p_a \neq 0$$

Przykłady

1°) rura obciążona tylko ciśnieniem wewnętrznym

$p_a = p$, $N = 0$, $p_b = 0$

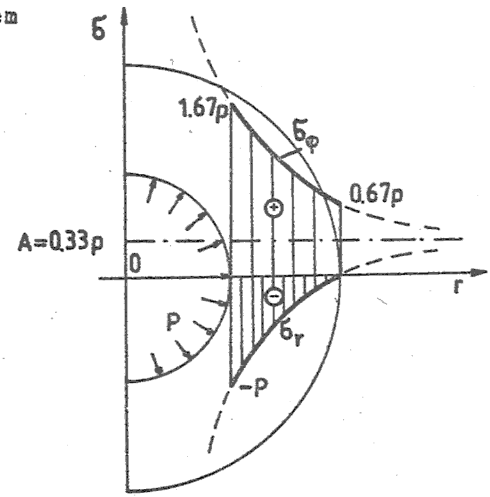
$$\frac{a}{b} = \frac{1}{2}$$

$$\sigma_r = p_a \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right)$$

$$\sigma_\varphi = p_a \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right)$$

$$\sigma_z = 0$$

$$A = p_a \frac{0{,}5^2}{1 - 0{,}5^2} = 0{,}33\, p$$



rys.6.

2°) cylinder obciążony tylko ciśnieniem zewnętrznym

$p_b = p \quad p_a = 0$

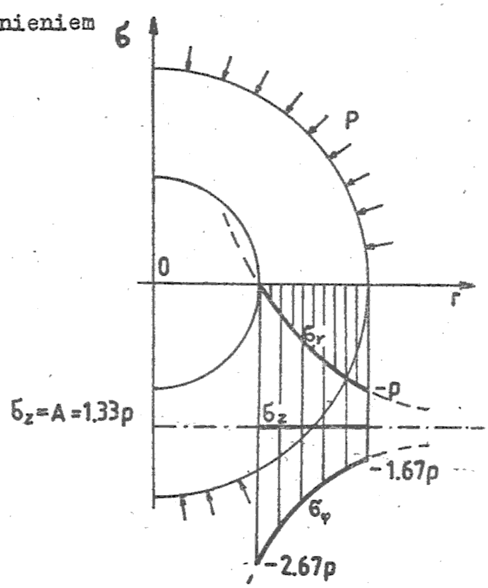
$$\sigma_r = -p_b \frac{b^2}{b^2 - a^2} \left(1 - \frac{a^2}{r^2} \right)$$

$$\sigma_\varphi = -p_b \frac{b^2}{b^2 - a^2} \left(1 + \frac{a^2}{r^2} \right)$$

$$\sigma_z = \frac{N}{F} = -p_b \frac{b^2}{b^2 - a^2} = A$$

dla $\frac{a}{b} = 0{,}5$

$$A = -p_b \frac{1}{1 - 0{,}5^2} = -1{,}333p$$



rys.7.

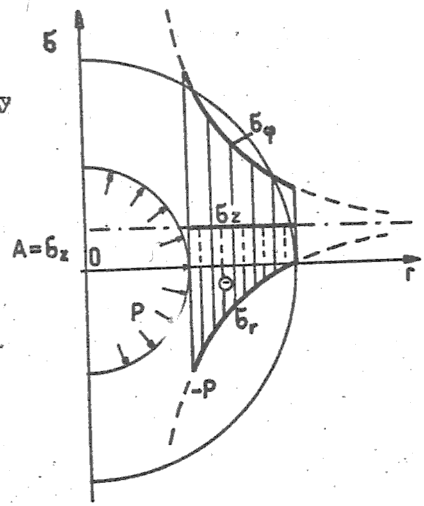
3°) cylinder zamknięty dnami obciążony ciśnieniem wewnętrznym

$$\sigma_r = p_a \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma = p_a \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_z = \frac{N}{F} \quad \frac{p_a \pi a^2}{\pi(b^2 - a^2)} =$$

$$= p_a \frac{a^2}{b^2 - a^2} = A$$



rys.8.

2. Wytężenie cylindrów grubościennych

hip. H.M.H.

(18) $$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{(\sigma_r - \sigma_\varphi)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\varphi - \sigma_z)^2}$$

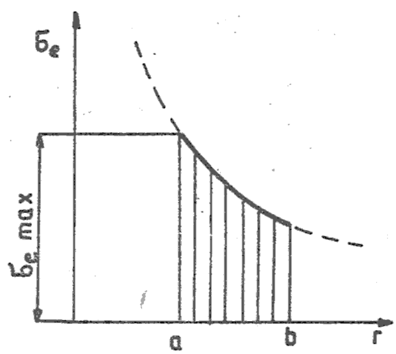
$$\sigma_r(r) = A - B \frac{1}{r^2}$$

$$\sigma_\varphi(r) = A + B \frac{1}{r^2}$$

$$\sigma_z = \frac{N}{F}$$

(19) $$\sigma_e(r) = \sqrt{(A - \sigma_z)^2 + 3 B^2 \frac{1}{r^4}}$$

$$\sigma_e(r)\Big|_{r=a} = \sigma_0 \max$$



rys.9.

Warunek bezpieczeństwa :

(20) $\sigma_{e\,max} \leqslant k_r$

(20a) $\sqrt{(A - \sigma_z)^2 + 3\,B^2\,\frac{1}{a^2}} \leqslant k_r$

hip. Burzyńskiego (przybliżenie paraboliczne)

(21) $$\sigma_e = \frac{1}{2Z}\Big[(Z-1)(\sigma_r + \sigma_\varphi + \sigma_z) +$$

$$+\sqrt{(Z-1)^2(\sigma_r + \sigma_\varphi + \sigma_z)^2 + 4Z(\sigma_r^2 + \sigma_\varphi^2 + \sigma_z^2 - \sigma_r\sigma_\varphi - \sigma_r\sigma_z - \sigma_\varphi\sigma_z)}\Big]$$

gdzie :

$Z = k_c/k_r$ (nie mylić z osią z)

(22) $$\sigma_e(r) = \frac{1}{2Z}\left\{(Z-1)(2A+\sigma_z) + \sqrt{(Z-1)^2(2A+\sigma_z)^2 + 4Z\left[(A-\sigma_z)^2 + 3B^2\frac{1}{r^4}\right]}\right\}$$

w.b.

(23) $$\sigma_{e\,max} = \sigma_e(r)\Big|_{r=a} \leqslant k_r$$

hip. TG

(24) $$\sigma_e = \sigma_I - \sigma_{III}$$

np. dla przypadku z przykładu 3°) (str. 9)

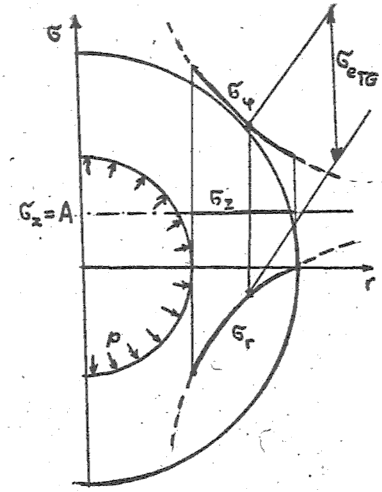
$\sigma_I = \sigma_\varphi$

$\sigma_{II} = \sigma_z$

$\sigma_{III} = \sigma_r$

$\sigma_{e\ TG} = \sigma_\varphi - \sigma_r$

(odległość hiperbol.)



rys.10

Przykład :

Znaleźć kres górny dopuszczalnego ciśnienia wewnętrznego dla rury grubościennej ($p_b = 0$, $\sigma_z = 0$).

$p_a = p$

$\frac{p_b}{p_a} = t \qquad t = 0$ bo $p_b = 0$

p będzie maksymalne gdy $b \to \infty$

$\frac{a}{b} = n \to 0$

$$A = p \frac{n^2 - t}{1 - n^2} = 0$$

$$B = p\, a^2 \frac{1 - t}{1 - n} = p \cdot a^2$$

warunek bezpieczeństwa :

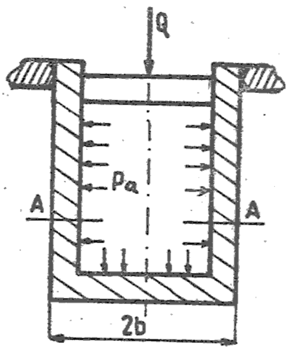
$\sigma_{e\ max} \leqslant k_r \quad /(\sigma_{e\ max} = \sigma_e (a)/$

$$\sqrt{3\, p\, a^2 \, \frac{1}{a^2}} \leqslant k_r \longrightarrow p \leqslant \frac{k_r}{3}$$

$$p_{dop} \leqslant 0{,}577\, k_r$$

Przykład :

Znaleźć grubość ścianki dla cylindra obciążonego jak na rys.



rys.11.

Dane :

$a = 4{,}0$ cm

$p_a = p = 2500$ MPa

Szukane :

$b = ?$

lub

$\frac{a}{b} = n = ?$

$$t = \frac{p_b}{p_a} = 0$$

$$\sigma_z \, \pi \, (b^2 - a^2) = p \pi a^2 \rightarrow \sigma_z = \frac{p\, a^2}{b^2 - a^2} = p \, \frac{n^2}{1 - n^2}$$

z (17 a) jest :

$$A = p \, \frac{n^2}{1 - n^2} \quad , \quad B = p \cdot a^2 \, \frac{1}{1 - n^2}$$

z. (19) :

$$\sigma_{e\,max} = \sqrt{\left(p\,\frac{n^2}{1-n^2} - p\,\frac{n^2}{1-n^2}\right)^2 + 3p^2\left(\frac{1}{1-n^2}\right)^2 \frac{a^4}{a^4}} =$$

$$= \frac{p\sqrt{3}}{(1-n^2)}$$

Warunek bezpieczeństwa :

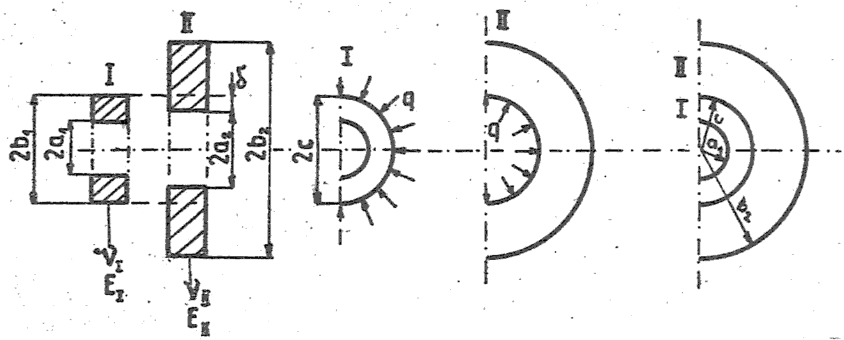
$$\sigma_{e\,max} \leqslant k_r \longrightarrow \frac{p\sqrt{3}}{1-n^2} \leqslant k_r$$

$$\frac{p\sqrt{3}}{k_r} \leqslant 1 - n^2 \longrightarrow n^2 \leqslant 1 - \frac{p\sqrt{3}}{k_r}$$

$$b \geqslant \frac{a}{\sqrt{1 - \sqrt{3}\,\frac{p}{k}}}$$

$$b \geqslant \frac{4\ \text{cm}}{1-\sqrt{3}\ .\ 0{,}5} = 10{,}9\ \ \text{cm}$$

3. Cylindry wielowarstwowe



rys. 1.2.

$$\delta = b_1 - a_2 \qquad a_2 = b_1 - \delta = b_1 \left(1 - \frac{\delta}{b_1}\right)$$

(25)

$$a_2 \cong b_1 \cong c$$

(26)

$$\delta = \left|u_{I_{r=b_1}}\right| + \left|u_{II_{r=a_2}}\right|$$

na podstawie (13), (15) gdy :

$$p_a = 0 \,, \; p_b = q \,, \; N = 0$$

$$u_{I_{r=b_1}} = \frac{1}{2G_1(b_1^2 - a_1^2)}\left[(1 - 2\nu_I)(-q \cdot b_1^2) \cdot b_1 + a_1^2 b_1^2 (-q) \frac{1}{b_1}\right] -$$

$$- \frac{2\nu_I^2}{E_I} \frac{q \cdot b_1^3}{b_1^2 - a_1^2}$$

po przekształceniach :

$$\boxed{u = C_1 r + \frac{C_2}{r}} = \left(\frac{1-2\nu}{2G} A - \nu \varepsilon_z\right) r + \frac{1}{2G} B \frac{1}{r}$$

$$\varepsilon_z = -\frac{2\nu}{E} A + \frac{1}{E} \sigma_z$$

(27)

$$\left| u_{I\,r=b_1} \right| = \frac{b_1}{E_I}\left(\frac{1+n_1^2}{1-n_1^2} - \nu_I\right) q$$

gdzie : $n_1^2 = \frac{a_1^2}{b_1^2}$

analogicznie dla II gdy :

$p_a = q\ ,\ p_b = 0\ ,\ N = 0$

(28)

$$\left| u_{II\ r=a_2} \right| = \frac{a_2}{E_{II}}\left(\frac{1+n_2^2}{1-n_2^2} + \nu_{II}\right) \cdot q$$

$n_2^2 = \frac{a_2^2}{b_2^2}$

po uwzględnieniu (25), (27) + (28):

(29) $$\delta = \frac{c}{E_I}\left(\frac{1+n_1^2}{1-n_1^2} - \nu_I\right) q + \frac{c}{E_{II}}\left(\frac{1+n_2^2}{1-n_2^2} + \nu_{II}\right) q$$

(30)

$$q = \frac{\delta}{\frac{c}{E_I}\left(\frac{1+n_1^2}{1-n_1^2} - \nu_I\right) + \frac{c}{E_{II}}\left(\frac{1+n_2^2}{1-n_2^2} + \nu_{II}\right)}$$

gdy $E_I = E_{II}$ $\nu_I = \nu_{II}$

$$q = \frac{\delta E}{2c}\ \frac{(1-n_1^2)(1-n_2^2)}{(1+n_1^2)(1-n_2^2) + (1+n_2^2)(1-n_1^2)}$$

Rozkład naprężeń $- \sigma_{\substack{r\\ \varphi}} = A \mp \frac{B}{r^2}$

a) ciśnienie wcisku q

cylinder I :

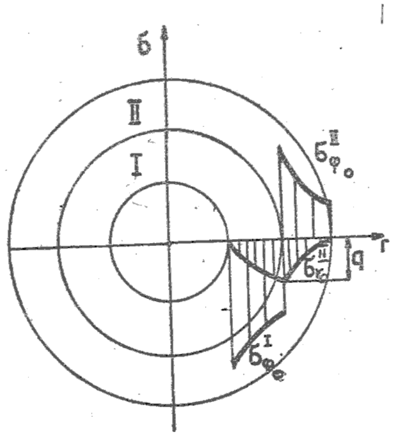
$\left.\begin{array}{l}\sigma_r^I(a_1) = 0 \\ \sigma_r^I(c) = -q\end{array}\right\} A^{Iq}, B^{Iq}$

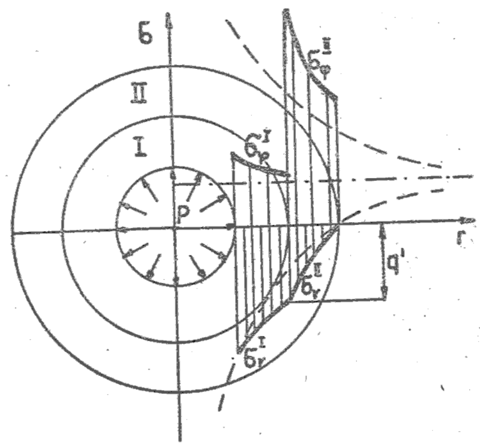
cylinder II:

$\left.\begin{array}{l}\sigma_r(c) = -q \\ \sigma_r^{II}(b_2) = 0\end{array}\right\} A^{IIq}, B^{IIq}$

b) ciśnienie robocze p i wcisk q

$\left.\begin{array}{l}\sigma_r(a_1) = -p \\ \sigma_r^I(c) = -q' \\ \sigma_r^{II}(c) = -q' \\ \sigma_r^{II}(b_2) = 0 \\ u_I(c) = u_{II}(c)\end{array}\right\} \begin{array}{l}A^I, B^I \\ A^{II}, B^{II} \\ q'\end{array}$





Warunek wyrównania maksymalnego wytężenia (optymalności) :

$$\sigma_{e_{max}}^{I} = \sigma_{e_{max}}^{II} \longrightarrow \sigma_e^I(a_1) = \sigma_e^{II}(c)$$

$$\sqrt{\left(A^I\right)^2 + \frac{3}{a_1^4}\left(B^I\right)^2} = \sqrt{\left(A^{II}\right)^2 + \frac{3}{c^4}\left(B^{II}\right)^2} \dashrightarrow q_{opt} \text{ lub } c_{opt}$$

dla cylindrów z tego samego materiału $c_{opt.} = \sqrt{a_1 b_2}$

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